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Cosmological and black hole spacetimes in twisted noncommutative gravity

Thorsten Ohl and Alexander Schenkel

Institut für Theoretische Physik und Astrophysik, Universität Würzburg, Am Hubland, 97074 Würzburg, Germany
E-mail: ohl@physik.uni-wuerzburg.de, aschenkel@physik.uni-wuerzburg.de

Abstract: We derive noncommutative Einstein equations for abelian twists and their solutions in consistently symmetry reduced sectors, corresponding to twisted FRW cosmology and Schwarzschild black holes. While some of these solutions must be rejected as models for physical spacetimes because they contradict observations, we find also solutions that can be made compatible with low energy phenomenology, while exhibiting strong noncommutativity at very short distances and early times.

Keywords: Non-Commutative Geometry, Models of Quantum Gravity, Space-Time Symmetries

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Contents

1 Introduction

Despite the great success of Einstein's general theory of relativity, it is generally believed that it has to be modified at small distances, incorporating quantum effects of spacetime. To achieve this goal and arrive at a consistent theory of quantum gravity, a number of different approaches have been proposed, including string theory and loop quantum gravity as prominent examples. The aim of these models is to provide a microscopic description of quantum spacetime subsequently to make contact to more macroscopic phenomena, like e. g. our universe. In doing so it turns out that it is quite hard to connect the very small length scales on which these models are defined with the large scales on which observable physics takes place, e. g. cosmic inflation or particle physics.

A complementary approach towards quantum gravity is to construct effective theories as an intermediate step between general relativity and a full theory of quantum gravity and study physical applications within it. These results can then possibly be used to connect full quantum gravities to physical phenomena. There have been many approaches in this direction, from which we choose the approach of noncommutative (NC) gravity based on a modification of symmetries [1, 2] (see also the review [3]). The main idea behind this formalism is to replace the classical symmetries of general relativity (i.e. the diffeomorphisms) by a twist deformed Hopf algebra of diffeomorphisms, which can be interpreted as quantum symmetries. As a result of this postulate one obtains that these theories naturally live on noncommutative spacetimes.

For effective theories, it is crucial to find solutions and study the physics they describe. For the case of the gravity theory proposed in [1, 2] the only solutions known are the NC black hole models discussed in [4]. This lack of models provided the motivation for us to systematically discuss symmetry reduction in Hopf algebra based gravity theories in our previous paper [5]. This can be seen as a first step towards the construction of solutions. We found that there are consistency conditions restricting compatible Drinfel'd twists for a given symmetry. This has lead to a classification of admissible deformations of Friedmann-Robertson-Walker universes and Schwarzschild black holes in the presence of a certain type of twist deformations, the so-called abelian or Reshetikhin-Jambor-Sykora (RJS) twists [6, 7].

The main goal of this brief paper is to show that most of our models solve the noncommutative Einstein equations proposed in [2]. Furthermore, some physical implications of our models are discussed. Therefore, we briefly review the models proposed in [5] in section 2 and discuss the phenomenologically interesting models. In section 3 we briefly review the noncommutative Riemannian geometry and Einstein equations of [2]. We will give a simplified formalism for the case of RJS twists by using a special basis of vector fields and one-forms on the manifold. In section 4 and 5 we show that most of our models solve the noncommutative Einstein equations and discuss some phenomenology. In section 6 we conclude and give an outlook to possible future investigations in this field.

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2 Review of our cosmological and black hole models

In our previous paper [5] we have classified possible noncommutative Friedmann-Robertson-Walker (FRW) cosmologies and Schwarzschild black holes in the presence of a Reshetikhin-Jambor-Sykora (RJS) twist [6, 7]. These twists are given by

$$\mathcal{F}_V := \exp\left(-\frac{i\lambda}{2}\Theta^{\alpha\beta}V_\alpha\otimes V_\beta\right)~, \tag{2.1}$$

where $\{V_\alpha\in\Xi\}$ is a set of mutually commuting vector fields and $\Theta^{\alpha\beta}$ can be taken in the canonical (i.e. Darboux) form.

Using the $\star$-commutators $[x^\mu \stackrel{\star}{,} x^\nu]$ among the linear coordinate functions from the appendix of [5], we can restrict our models to physically sensible cases. As a criterion for the cosmological models $\mathfrak{C}_{AB}$ we demand that the scale of noncommutativity does not grow in physical length scales. This excludes in particular the Moyal-Weyl type model $\mathfrak{C}_{11}$ with $\left[x^i \stackrel{\star}{,} x^j\right]=i\theta^{ij}$, where $\theta^{ij}=\text{const.}$, since x^i are comoving spatial coordinates and have to be multiplied by the scale factor of the universe $A(t)$ in order to give physical length scales. Since the scale factor grows rapidly during inflation, the physical scale of noncommutativity $A(t)^2\theta^{ij}$ grows too, leading to a very noncommutative late universe, which contradicts observations. Including analogous arguments for the time-space $\star$-commutators we obtain the physically valid models $\mathfrak{C}_{22}$ with $\mathbf{c}_2=0$ or $|V_1^0(t)A(t)|$ nongrowing in t and $\mathfrak{C}_{32}$ with $\mathbf{c}_2=0$. The vector fields V_α generating these models are given by

$$\mathfrak{C}_{22}: \qquad V_1 = V_1^0(t)\partial_t~,\quad V_2 = c_2^i\partial_i + d_2^iL_i + f_2x^i\partial_i~, \tag{2.2a}$$

$$\mathfrak{C}_{32}: \qquad V_1 = V_1^0(t)\partial_t + d_1^iL_i~,\quad V_2 = V_2^0(t)\partial_t + f_2x^i\partial_i~. \tag{2.2b}$$

Here $L_i := \epsilon_{ijk}x^j\partial_k$ are the generators of rotations and $\left[V_1^0(t)\partial_t, V_2^0(t)\partial_t\right]\equiv 0$. Note that we have switched the labels of the vector fields V_α in model $\mathfrak{C}_{22}$ compared to [5] for later convenience. Furthermore, we will restrict ourselves to the case $\mathbf{c}_2 = 0$ for the model $\mathfrak{C}_{22}$ for the following reason: the case $\mathbf{c}_2 \neq 0$ requires a rapidly decreasing $V_1^0(t)$ for an inflationary scenario. Thus the model can be well approximated by the model $\mathfrak{C}_{22}$ with $d_2^i = f_2 = 0$, since the additional terms will be suppressed by $V_1^0(t)$ in physical coordinates. The resulting model is then simply a time-space Moyal deformation, which has been discussed elsewhere. We will omit an explicit discussion of this model for brevity and only note that it can be described by the methods developed below as well.

As a physicality criterion for our Schwarzschild black hole models $\mathfrak{B}_{AB}$ [5] we use the requirement $N_1 = N_2 = 0$, since otherwise noncommutativity would grow linearly in time. This leads to the physically viable model $\mathfrak{B}_{12}$ constructed by the vector fields

$$\mathfrak{B}_{12}: \qquad V_1 = c_1^0\partial_t + \kappa_1 d^i L_i\,, \quad V_2 = c_2^0(r)\partial_t + \kappa_2 d^i L_i + f_2(r)x^i\partial_i\,, \tag{2.2c}$$

where c_1^0 has to be constant and $r := \|\mathbf{x}\|$ is the radial coordinate. Note that the other physically viable model $\mathfrak{B}_{11}$ is already included in the class $\mathfrak{B}_{12}$.

We can understand our models better by choosing without loss of generality $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{d} = (0,0,d)$ and transforming from cartesian coordinates x^i to spherical coordinates (r,ζ,ϕ). Then the vector fields read

$$\mathfrak{C}_{22}: \qquad V_1 = V_1^0(t)\partial_t\,, \quad V_2 = d\partial_\phi + f_2 r\partial_r\,, \tag{2.3a}$$

$$\mathfrak{C}_{32}: \qquad V_1 = V_1^0(t)\partial_t + d\partial_\phi\,, \quad V_2 = V_2^0(t)\partial_t + f_2 r\partial_r\,, \tag{2.3b}$$

$$\mathfrak{B}_{12}: \qquad V_1 = c_1^0\partial_t + \kappa_1\partial_\phi\,, \quad V_2 = c_2^0(r)\partial_t + \kappa_2\partial_\phi + f(r)\partial_r\,. \tag{2.3c}$$

Note that we have defined $f(r) := f_2(r)r$ and absorbed the parameter d into κ_α in the model $\mathfrak{B}_{12}$ in order to simplify the expression.

The $\star$-commutation relations among appropriate coordinate functions in spherical coordinates are:

$$\mathfrak{C}_{22}: \begin{cases} [t \overset{\star}{,} \exp i\phi] = -2\exp i\phi\ \sinh\left(\dfrac{\lambda d}{2}V_1^0(t)\partial_t\right)t \\ [t \overset{\star}{,} r] = 2ir\ \sin\left(\dfrac{\lambda f_2}{2}V_1^0(t)\partial_t\right)t \end{cases} \tag{2.4a}$$

$$\mathfrak{C}_{32}: \begin{cases} [t \overset{\star}{,} \exp i\phi] = 2\exp i\phi\ \sinh\left(\dfrac{\lambda d}{2}V_2^0(t)\partial_t\right)t \\ [t \overset{\star}{,} r] = 2ir\ \sin\left(\dfrac{\lambda f_2}{2}V_1^0(t)\partial_t\right)t \\ \exp i\phi \star r = e^{-\lambda d f_2}\ r \star \exp i\phi \end{cases} \tag{2.4b}$$

$$\mathfrak{B}_{12}: \begin{cases} [t \overset{\star}{,} \exp i\phi] = \exp i\phi\ \left(2\sinh\left(\dfrac{\lambda\kappa_1}{2}\left(c_2^0(r)\partial_t + f(r)\partial_r\right)\right)t - \lambda\kappa_2 c_1^0\right) \\ [t \overset{\star}{,} r] = i\lambda c_1^0 f(r) \\ [\exp i\phi \overset{\star}{,} r] = -2\exp i\phi\ \sinh\left(\dfrac{\lambda\kappa_1}{2}f(r)\partial_r\right)r \end{cases} \tag{2.4c}$$

In particular, our models include time-angle, time-radius and angle-radius noncommutativity for both the FRW cosmologies and the Schwarzschild black holes. Note further that the $\star$-commutators simplify dramatically for the choice $V_\alpha^0(t) = \text{const.}$, $f(r) = r$ and $c_2^0(r) = \text{const.}$. This will be further explained below, when we discuss specific examples.

3 Review of twisted noncommutative Einstein equations

In this section we will briefly review the noncommutative Riemannian geometry and Einstein equations constructed in [2] (see also [1] and [3]). We will restrict the discussion to RJS twists (2.1).

Since the formulae in [2] were constructed in a coordinate and basis independent way, we have the freedom to choose a suitable basis for the vector fields $\{e_a \in \Xi : a = 0, \dots, 3\}$ and one-forms $\{\theta^a \in \Omega : a = 0, \dots, 3\}$. It turns out that the expressions for the geometrical quantities and the Einstein equations do simplify drastically, if we can find a basis of vector fields $\{e_a\}$ satisfying

$$[e_a, e_b] = 0\,, \quad [V_\alpha, e_a] = 0\,, \tag{3.1}$$

for all a, b, α. We call the basis (3.1) the natural basis of vector fields and construct the basis of one-forms $\{\theta^a\}$ by duality

$$\delta_a^b = \left\langle e_a, \theta^b \right\rangle_\star = \left\langle \bar{f}^\alpha(e_a), \bar{f}_\alpha(\theta^b) \right\rangle = \left\langle e_a, \theta^b \right\rangle\,, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ is the canonical commutative pairing between vector fields and one-forms and $\bar{f}^\alpha \otimes \bar{f}_\alpha = \mathcal{F}_V^{-1}$ is the inverse twist.

The existence of a local (densely defined) natural basis (3.1) can be shown explicitly for the case of RJS twists (2.1), assuming that the vector fields V_α are analytical almost everywhere. In this brief paper we will omit the general proof and only give the natural basis for our explicit models (2.3). It turns out that for the cosmological models $\mathfrak{C}_{22}$ and $\mathfrak{C}_{32}$ we can make the choice

$$e_0 = v(t)\partial_t\,, \quad e_1 = r\partial_r\,, \quad e_2 = \partial_\zeta\,, \quad e_3 = \partial_\phi\,, \quad \text{where} \tag{3.3a}$$

$$v(t) = \begin{cases} V_1^0(t) & \text{, for model } \mathfrak{C}_{22} \text{ and } \mathfrak{C}_{32} \text{ with } V_1^0(t) \not\equiv 0 \\ V_2^0(t) & \text{, for model } \mathfrak{C}_{32} \text{ with } V_1^0(t) \equiv 0,\ V_2^0(t) \not\equiv 0 \\ 1 & \text{, for model } \mathfrak{C}_{32} \text{ with } V_1^0(t) \equiv V_2^0(t) \equiv 0\ . \end{cases} \tag{3.3b}$$

For the black hole model $\mathfrak{B}_{12}$ we have to discuss the cases $f(r) \not\equiv 0$ and $f(r) \equiv 0$ separately. For the first case we can use

$$e_0 = \partial_t\,, \quad e_1 = f(r)\partial_r + c_2^0(r)\partial_t\,, \quad e_2 = \partial_\zeta\,, \quad e_3 = \partial_\phi\,, \tag{3.4}$$

as a natural basis. For the second case, the twist vector fields can be reduced without loss of generality to $V_1 = \kappa_1 \partial_\phi$ and $V_2 = c_2^0(r)\partial_t$, such that a natural basis would be

$$e_0 = c_2^0(r)\partial_t\,, \quad e_1 = \partial_r + t c_2^0(r)'/c_2^0(r)\partial_t\,, \quad e_2 = \partial_\zeta\,, \quad e_3 = \partial_\phi\,, \tag{3.5}$$

where $c_2^0(r)'$ denotes the derivative of $c_2^0(r)$. It can be checked directly that (3.1) is satisfied.

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Next, we consider a metric field $g = \theta^a \otimes_\star \theta^b \star g_{ba} = \theta^a \otimes \theta^b g_{ba}$ in the natural basis. Note that the $\star$-tensor product and $\star$-product in the expression above reduce to the undeformed products, since the twist acts trivially on the basis one-forms $\{\theta^a\}$. Furthermore, we have $g_{ab} = g_{ba}$. The inverse metric $g^{-1} = g^{ab} \star e_b \otimes_\star e_a = g^{ab} e_b \otimes e_a$ defined in [2] satisfies in the natural basis

$$g_{ab} \star g^{ca} = g^{ac} \star g_{ba} = \delta_b^c \, , \tag{3.6}$$

i.e. it is given by the $\star$-inverse matrix of g_{ab}.

The $\star$-covariant derivative on tensor fields was defined in [2]. Using the natural basis (3.1) we obtain for its basis representation

$$(\nabla^\star_{e_c} \tau)^{a_1 \dots a_n}_{b_1 \dots b_l} = e_c \left(\tau^{a_1 \dots a_n}_{b_1 \dots b_l} \right) - \Gamma_{cb_1}{}^{\tilde{b}} \star \tau^{a_1 \dots a_n}_{\tilde{b} \dots b_l} - \dots + \tau^{a_1 \dots \tilde{a}}_{b_1 \dots b_l} \star \Gamma_{c\tilde{a}}{}^{a_n} \, . \tag{3.7}$$

Here $\Gamma_{ab}{}^c \star e_c := \nabla^\star_{e_a} e_b$ are the connection symbols and $e_c(\cdot)$ is the vector field action on functions (Lie derivative).

In the natural basis the torsion tensor $\mathrm{T} = \theta^b \otimes \theta^a \mathrm{T}_{ab}{}^c \otimes e_c$ defined in [2] reduces to

$$\mathrm{T}_{ab}{}^c = \Gamma_{ab}{}^c - \Gamma_{ba}{}^c \, . \tag{3.8}$$

The metric compatible torsionfree connection is given by

$$\Gamma_{ab}{}^c = \frac{1}{2} \big(e_a(g_{bd}) + e_b(g_{ad}) - e_d(g_{ab}) \big) \star g^{cd} \, . \tag{3.9}$$

In the natural basis the expression for the Riemann tensor $\mathrm{R} = \theta^c \otimes \theta^b \otimes \theta^a \mathrm{R}_{abc}{}^d \otimes e_d$ simplifies to

$$\mathrm{R}_{abc}{}^d = e_a \left(\Gamma_{bc}{}^d \right) - e_b \left(\Gamma_{ac}{}^d \right) + \Gamma_{bc}{}^e \star \Gamma_{ae}{}^d - \Gamma_{ac}{}^e \star \Gamma_{be}{}^d \, . \tag{3.10}$$

The Ricci tensor is given by $\mathrm{Ric}_{ab} = \mathrm{R}_{cab}{}^c$ and the curvature scalar is given by $\mathcal{R} = g^{ab} \star \mathrm{Ric}_{ba}$.

This leads to the NC Einstein equations proposed in [2]

$$G_{ab} := \mathrm{Ric}_{ab} - \frac{1}{2} g_{ab} \star \mathcal{R} = M_{pl}^{-2} T_{ab} \, , \tag{3.11}$$

where we have introduced the Einstein tensor G_{ab}, the Planck mass M_{pl} and a stress-energy tensor field T_{ab}. In this work we only need to assume that T_{ab} is constructed from some (scalar) matter field ϕ and its covariant derivatives in a deformed covariant way. We assume further that the stress-energy tensor is at least quadratic in the matter fields.

Note that in the natural basis all geometrical quantities defined above include only $\star$-products among the *coefficient functions* of tensor fields, and not among the basis vector fields and one-forms. Thus the formalism in the natural basis is more convenient for doing explicit calculations than the basis independent formalism of [2, 3].

To conclude this section we will briefly discuss possible issues with the NC Einstein equations (3.11). Firstly, it is not necessarily a real tensor field and secondly, the right hand side of the contracted second Bianchi identity

$$g^{ba} \star (\nabla^\star_{e_a} G)_{cb} = \frac{\Delta_c}{2} \, , \tag{3.12}$$

does not vanish, where

$$\Delta_c = g^{ba} \star \left((\nabla^\star_{e_a} \mathrm{Ric})_{cb} - (\nabla^\star_{e_d} \mathrm{R})_{cab}{}^d \right) - g^{ba} \star \left(\left[\Gamma_{cd}{}^{\tilde{d}} \stackrel{\star}{,} \mathrm{R}_{\tilde{d}ab}{}^d \right] - \left[\Gamma_{ad}{}^{\tilde{d}} \stackrel{\star}{,} \mathrm{R}_{\tilde{d}cb}{}^d \right] \right)$$
$$- \left(\nabla^\star_{e_c} g^{-1} \right)^{ba} \star \mathrm{Ric}_{ab} \; . \tag{3.13}$$

The first issue is not too dramatic and can in principle be solved by adding the complex conjugate tensor, but the second issue in general leads to problems when coupling matter to gravity. In this case the stress-energy tensor would have to satisfy

$$g^{ba} \star \left(\nabla^\star_{e_a} T \right)_{cb} = \frac{\Delta_c}{2} \, , \tag{3.14}$$

which is in general not compatible with sensible equations of motion for the matter fields for the case $\Delta_c \neq 0$.

To solve this issue one could try to define a modified Einstein tensor $\tilde{G}_{ab}$, such that

$$g^{ba} \star \left(\nabla^\star_{e_a} \tilde{G} \right)_{cb} = 0 \; . \tag{3.15}$$

This could possibly be done for explicit problems by integrating the right hand side of (3.12). Fortunately, it will turn out that for most of our models (2.3) these problems do not occur and we find $\tilde{G}_{ab} = G_{ab}$. Because of this we postpone the issue of modifying the Einstein tensor to a future work and only discuss our well defined solutions in the following sections.

4 Cosmological solutions

Firstly, we discuss exact cosmological solutions of the NC Einstein equations (3.11). For this we can use proposition 4 of [5] in order to find the right ansatz for the symmetry reduced metric field g. This proposition tells us that a tensor field is invariant under the deformed action of the deformed symmetries, if and only if it is invariant under the undeformed action of the undeformed symmetries. The requirement for this proposition was to use the so-called canonical embedding of the symmetry Lie algebra given by $\mathfrak{g}_\star = \mathfrak{g}$, which is fulfilled for our models defined in section 2. In [5] we have classified the compatible twists for spatially flat FRW cosmologies and we will restrict ourselves to this case here as well. There is however no principal obstruction to generalising the classification in [5] to spatially curved cosmologies and to constructing the corresponding solutions. We make the ansatz $g = dx^\mu \otimes dx^\nu g_{\nu\mu}$ in the commutative coordinate basis, with

$$g_{\mu\nu} = \mathrm{diag}\left(-1, A(t)^2, A(t)^2, A(t)^2 \right)_{\mu\nu} \, , \tag{4.1}$$

and calculate the required coefficients g_{ab} in the natural basis by solving

$$\theta^a \otimes \theta^b g_{ba} = dx^\mu \otimes dx^\nu \theta^a_\mu \theta^b_\nu g_{ba} = dx^\mu \otimes dx^\nu g_{\nu\mu} \, , \tag{4.2}$$

using the explicit expression of the natural basis vector fields (3.3).

It can be checked explicitly that for the choice $f_2 = 0$ in model $\mathfrak{C}_{22}$ (2.3a) and the choice $V_1^0(t) \equiv 0$ in model $\mathfrak{C}_{32}$ (2.3b) the NC connection symbols (3.9), the NC Riemann

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tensor (3.10) and finally the NC Einstein tensor (3.11) receive no contributions in the deformation parameter λ, thus reducing to the undeformed counterparts. The reason for this is that for the restrictions mentioned above we have one twist vector field $V_\alpha \in \mathfrak{g}$ and therefore deformed operations among symmetric tensors reduce to the undeformed ones, since $V_\alpha \in \mathfrak{g}$ annihilates the tensors due to invariance.

Since the NC stress-energy tensor of symmetric matter reduces to the undeformed tensor due to the same reasons, these NC models are exactly solvable, iff the undeformed model is exactly solvable. Note that the reduction of the deformed symmetric tensors to the undeformed ones does not mean that our models are trivial. In particular, we will obtain in general a deformed dynamics for fluctuations on the symmetry reduced backgrounds, as well as a nontrivial coordinate algebra (2.4).

Next, we will discuss physical implications of the nontrivial coordinate algebras of our models. Consider the model $\mathfrak{C}_{22}$ (2.3a) with $f_2 = 0$ and for simplicity $V_1^0(t) \equiv 1$. Then the coordinate algebra (2.4a) reduces to the algebra of a quantum mechanical particle on the circle, i.e.

$$\left[\hat{E}, \hat{t}\right] = \lambda \hat{E}\,, \tag{4.3}$$

where we introduced the abstract operators $\hat{t}$ and $\hat{E} := \widehat{\exp i\phi}$ and set $d = 1$. This algebra previously appeared e. g. in the context of noncommutative field theory [8] and the noncommutative BTZ black hole [9]. It is well known that the operator $\hat{t}$ can be represented as a differential operator acting on the Hilbert space $L^2(S_1)$ of square integrable functions on the circle and the spectrum can be shown to be given by $\sigma(\hat{t}) = \lambda(\mathbb{Z} + \delta)$, where $\delta \in [0, 1)$ labels unitary inequivalent representations. The spectrum should be interpreted as possible time eigenvalues. Thus our model with discrete time can be used to realize singularity avoidance in cosmology. Consider for example an inflationary background with $A(t) = t^p$, where $p > 1$ is a parameter. This so-called power-law inflation can be realized by coupling a scalar field with exponential potential to the geometry even in our NC model, since the symmetry reduced Riemannian geometry reduces to the undeformed one as explained above. Note that the scale factor goes to zero at the time $t = 0$ and leads to a singularity in the curvature scalar. But as we discussed above, the possible time eigenvalues are $\lambda(\mathbb{Z} + \delta)$, which does not include the time $t = 0$ for $\delta \neq 0$.

Note that the discrete time and therewith possible singularity avoidance is a general feature of models with time-angle noncommutativity, even if $V_1^0(t) \neq \text{const.}$. This can be seen by performing a local time reparametrization, such that $V_1^0(t) \equiv 1$, and pulling back the discrete spectrum of the time operator. The effect of this pullback is that the distance between the time eigenvalues will not be uniform in general.

For the more complicated solvable model $\mathfrak{C}_{32}$ (2.3b) with $V_1^0(t) \equiv 0$ we obtain time-angle and angle-radius noncommutativity. We set without loss of generality the parameter $d = f_2 = 1$. Firstly, we choose $V_2^0(t) \equiv 0$ leading to a pure angle-radius noncommutativity. The algebra (2.4b) becomes in this case

$$\hat{E}\hat{r} = e^{-\lambda}\, \hat{r}\hat{E}\ . \tag{4.4}$$

This algebra can be represented on the Hilbert space $L^2(S_1)$ as

$$\hat{E} = \exp i\phi\,, \quad \hat{r} = \Lambda \exp\left(-i\lambda\partial_\phi\right), \tag{4.5}$$

leading to the spectrum $\sigma(\hat{r}) = \Lambda\exp\bigl(\lambda(\mathbb{Z}+\delta)\bigr)$, where $\delta \in [0,1)$ is again a parameter labeling unitary inequivalent representations and Λ is some length scale. Therefore the radius becomes discrete. It would be natural to choose for Λ a physical length scale (e. g. the Hubble length) in order to avoid a growth of the eigenvalue spacings with time in an expanding universe. Note that this model describes a kind of "condensating geometry" around the origin $r = 0$, since the shells of constant r accumulate at this point. It remains to be shown for which values of λ and Λ this is consistent with experimental data from the cosmic microwave background (CMB).

Secondly, we choose for simplicity $V_2^0(t) \equiv 1$ in (2.4b). In the case of nonconstant $V_2^0(t)$ we can in principle pull back the spectrum as discussed before. We obtain the abstract algebra

$$\left[\hat{t}, \hat{E}\right] = \lambda\hat{E}\,, \quad \hat{E}\hat{r} = e^{-\lambda}\,\hat{r}\hat{E}\;. \tag{4.6}$$

Furthermore, we use the representation on $L^2(S_1)$

$$\hat{E} = \exp i\phi\,, \quad \hat{t} = \tau\hat{1} - i\lambda\partial_\phi\,, \quad \hat{r} = \Lambda\exp\left(-i\lambda\partial_\phi\right)\;. \tag{4.7}$$

Note that we had to introduce a real parameter $\tau \in \mathbb{R}$ and the identity operator $\hat{1}$ in order to cover the whole spacetime.

The last cosmological model we want to briefly discuss is the isotropic model $\mathfrak{C}_{22}$ with $d = 0$ (2.3a). It turns out that both $V_\alpha \notin \mathfrak{g}$, therefore the Riemannian geometry does not reduce to the undeformed one. Thus we expect corrections in λ to the NC Einstein equations (3.11) and its solutions. Since it is not yet clear how to formulate consistent NC Einstein equations coupled to matter, we postpone the investigation of these corrections to a future work and give here only one special exact solution of this model.

Consider the (undeformed) de Sitter space given by $A(t) = \exp Ht$, where H is the Hubble parameter. It turns out that all $\star$-products entering the deformed geometrical quantities (see section 3) reduce to the undeformed ones, if $V_1^0(t) \equiv 1$. Thus the undeformed de Sitter space solves NC Einstein equations (3.11), or possible modifications of it, for this particular choice of twist and an undeformed cosmological constant. Note that in contrast to the solutions above, we required the explicit form of the scale factor $A(t)$.

This demonstrates, by explicit construction, the existence of an exact solution of the isotropic model. It is natural to ask whether there are additional isotropic exact solutions. Constructing them or proving their existence or their absence must be the topic of future investigations. If this model is not exactly solvable for general matter, we could take an effective theory point of view and study solutions up to finite order in the deformation parameter λ. This way one could extract the leading NC effects on the dynamics of the scale factor.

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5 Black hole solutions

For the black hole model (2.3c) we use again proposition 4 of [5] and make the ansatz

$$g_{\mu\nu} = \mathrm{diag}\left(-Q(r), S(r), r^2, (r\sin\zeta)^2\right)_{\mu\nu} \tag{5.1}$$

for the metric field $g = dx^\mu \otimes dx^\nu g_{\nu\mu}$ in the commutative spherical coordinate basis. The metric in the natural basis can be calculated using (3.4) or (3.5), respectively. Concerning the solution of the NC Einstein equations we are in a comfortable position, since we have $V_1 \in \mathfrak{g}$, which means that the symmetry reduced Riemannian geometry reduces to the undeformed one. This leads in the exterior of our NC black hole to the metric (5.1) with

$$Q(r) = S(r)^{-1} = 1 - \frac{r_s}{r}\,, \tag{5.2}$$

where r_s is the Schwarzschild radius. As in the case of the cosmological models, the reduction of the symmetry reduced tensor fields to the undeformed counterparts does not mean that our models are trivial. Fluctuations (e.g. Hawking radiation), as well as the coordinate algebras will in general receive distinct NC effects.

Taking a look at the coordinate algebra of the black hole (2.4c), we observe that it includes in particular the algebra of a quantum mechanical particle on the circle for time and angle variable, if we choose $c_2^0(r) \equiv 0$ and $f(r) \equiv 0$. This leads to discrete times. Another simple choice is $c_1^0 = \kappa_2 = 0$, $c_2^0(r) \equiv 0$, $\kappa_1 = 1$ and $f(r) = r$. The radius spectrum in this case is $\sigma(\hat{r}) = \Lambda \exp\big(\lambda(\mathbb{Z}+\delta)\big)$, describing a fine grained geometry around the black hole. The phenomenological problem with this model is that the spacings between the radius eigenvalues grow exponentially in r. This can be fixed by considering a modified twist like e.g. $c_1^0 = \kappa_2 = 0$, $c_2^0(r) \equiv 0$, $\kappa_1 = 1$ and $f(r) = \tanh\frac{r}{\Lambda}$, where Λ is some length scale. The essential modification is to choose a bounded $f(r)$. Consider the coordinate change $r \to \eta = \log\sinh(\frac{r}{\Lambda})$. Note that r and not η is the physical radial coordinate of a Schwarzschild observer, η is just introduced in order to simplify the calculation. In terms of η the algebra (2.4c) reduces to

$$\left[\hat{E}, \hat{\eta}\right] = -\frac{\lambda}{\Lambda}\hat{E}\,, \tag{5.3}$$

leading to the spectrum $\sigma(\hat{\eta}) = \frac{\lambda}{\Lambda}\big(\mathbb{Z}+\delta\big)$. The spectrum of the physical radius $\hat{r}$ is then given by $\sigma(\hat{r}) = \Lambda\ \mathrm{arcsinh}\exp\big(\frac{\lambda}{\Lambda}(\mathbb{Z}+\delta)\big)$. This spectrum approaches constant spacings between the eigenvalues for large r.

We omit a deeper discussion of further possible models, since our main purpose was to present the very explicit and simple models shown above. We conclude this section with one remark. Our class of black hole models (2.3c) is related to the NC black hole models found earlier by Schupp and Solodukhin [4]. They also found that the symmetry reduced dynamics reduces to the undeformed one for their black hole models. In addition, they constructed models based on a projective twist, that is not contained in the RJS-class, which exhibit discrete radius eigenvalues as well.

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6 Conclusions and outlook

In this paper we have constructed exact cosmological and black hole solutions of the noncommutative gravity theory proposed in [1, 2]. In particular we have obtained FRW models in which the physical scale of noncommutativity is not growing with time, as it would be for the most simple Moyal-Weyl deformation. Some of our models possess interesting physical features, such as a discrete time spectrum for the case of FRW models and a discrete radius spectrum for the black hole. Furthermore, we have found that the most attractive cosmological model, deformed by an isotropic twist, solves the NC Einstein equations in presence of a cosmological constant. We also found that in particular for the isotropic twist FRW model, noncommutativity can in general influence the dynamics of the symmetry reduced sector. It will therefore be interesting to study, if we can use noncommutativity in order to drive inflation. In order to study these effects, one should modify the NC Einstein tensor as proposed in section 3 or use the recently proposed NC vielbein gravity [10] in order to couple matter and geometry properly.

In future work [12] we will construct scalar field fluctuations on curved NC backgrounds in a twisted covariant setting. This would also include the twist deformation of the Poisson algebra of field observables, as it was done in [11] for the case of the Moyal deformed Minkowski spacetime. Using this formalism, we will study NC modifications of the cosmic microwave background (CMB) and possibly also Hawking radiation. It will also be interesting to compare our approach with existing results on NC effects in the CMB, obtained in different settings [13].

Recently the physics of noncommutative Kerr black hole was studied [14] in the framework of [1]. It should be fruitful to study their results in our approach.

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